

Solving the Infeasible Trust-region Problem Using Approximations.

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I. Introduction

THE use of optimization in engineering design has fueled the development of algorithms for specific engineering needs. When the simulations are expensive to evaluate or the outputs present some noise, the direct use of nonlinear optimizers is not advisable, since the optimization process will be expensive and may result in premature convergence. The use of approximations for both cases is an alternative investigated by many researchers including the authors.

When approximations are present, a model management is required for proper convergence of the algorithm. In nonlinear programming, the use of trust-regions for globalization of a local algorithm has been proven effective. The same approach has been used to manage the local move limits in sequential approximate optimization frameworks as in Alexandrov *et al.*², Giunta and Eldred⁴, Pérez *et al.*⁶, Rodríguez *et al.*⁸, etc.

The experience in the mathematical community has shown that more effective algorithms can be obtained by the specific inclusion of the constraints (SQP type of algorithms) rather than by using a penalty function as in the augmented Lagrangian formulation. The presence of explicit constraints in the local problem bounded by the trust region, however, may have no feasible solution.

In order to remedy this problem the mathematical community has developed different versions of a composite steps approach. This approach consists of a normal step to reduce the amount of constraint violation and a tangential step to minimize the objective function maintaining the level of constraint violation attained at the normal step.

Two of the authors have developed a different approach for a sequential approximate optimization framework using homotopy ideas to relax the constraints. This algorithm called interior-point trust-region sequential approximate optimization (IPTRSAO) presents some similarities to the two normal-tangential steps algorithms. In this paper, a description of the similarities is presented and an expansion of the two steps algorithm is presented for the case of approximations.

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II. Sequential approximate optimization

The objective of a SAO algorithm is to solve optimization problems by the use of local approximations. Although the approximation can be of any type, second order polynomials are often used. An generalization of an optimization problem is the so called multidisciplinary design optimization problem. In this general case, the objective function and constraints are functions of the output states of a system simulation or simulation analysis (SA) composed by one or more single discipline analysis or contributing analysis (CA). The general multidisciplinary design optimization problem to be solved is of the form:

$$\begin{aligned} \min \quad & f(\mathbf{y}) \\ \text{s.t.} \quad & \mathbf{g}(\mathbf{y}) \geq 0, \\ & \mathbf{h}(\mathbf{y}) = 0, \\ & \mathbf{x}_{min} \leq \mathbf{x} \leq \mathbf{x}_{max}, \end{aligned} \tag{1}$$

where $\mathbf{y} = SA(\mathbf{x})$ is the vector of output states of the system analysis. From now on we will refer to the function and the constraints just as $f, \mathbf{g}, \mathbf{h}$ assuming they are functions of the states \mathbf{y} and therefore of the design variables \mathbf{x} . Also we will assume that all design variables are continuous.

The main idea of a sequential approximate optimization algorithm is to approximate locally the objective function and constraints by a response surface function, usually a quadratic polynomial. The approximation is constructed within the local region known as the trust region. The core of the algorithm is the solution of the local minimization subproblem. The solution of this problem yields a candidate design that is either accepted or rejected, and as a result the trust-region itself is updated.

A. Minimization Subproblem

Lets assume \mathbf{x}_0 is the starting design point, the trust region is defined and the response surface approximations of the objective function f and constraints \mathbf{h}, \mathbf{g} are available ($\tilde{f}, \tilde{\mathbf{h}}, \tilde{\mathbf{g}}$). The minimization subproblem can be written as:

$$\begin{aligned} \min \quad & \tilde{\theta} \\ \text{s.t.} \quad & \tilde{\mathbf{g}} \geq 0 \\ \text{s.t.} \quad & \tilde{\mathbf{h}} = 0 \\ & \mathbf{x}_L \leq \mathbf{x} \leq \mathbf{x}_U \end{aligned} \tag{2}$$

where θ is the objective function used for the minimization. The tilde refers to the response surface approximation of the function and \mathbf{x}_L and \mathbf{x}_U are the local variable bounds.

The inclusion of the approximate constraints $\tilde{\mathbf{g}} \geq 0$ and $\tilde{\mathbf{h}} = 0$ forces the local optimum to be located in the approximate feasible region, while the merit function $\tilde{\theta}$ guides the algorithm to the optimum. Different merit functions can be used. In the DAKOTA framework, $\theta = f$ while in the IPTRSAO algorithm θ is the augmented Lagrangian function as in Rockafellar⁹:

$$\theta = f + \boldsymbol{\lambda}_\psi^T \boldsymbol{\psi} + \boldsymbol{\lambda}_h^T \mathbf{h} + r \boldsymbol{\psi}^T \boldsymbol{\psi} + r \mathbf{h}^T \mathbf{h}, \tag{3}$$

where

$$\psi_i = \min\left\{g_i, -\frac{\lambda_i}{2r}\right\} \tag{4}$$

An SQP-like alternative is to make θ the Lagrangian function.

B. Infeasible Starting Point

In real engineering problems, it is almost impossible for the designer to provide a feasible starting point. If this is the case, the local subproblem may not have a feasible solution. Although the optimizer used to solve problem (2) may give a point that reduces the constraint violation, no robustness of the algorithm can be assured.

A simple alternative is the use of an initial algorithm that seeks a feasible design and once it is found, the algorithm would switch to solve the subproblem (2). However, this is not practical since seeking a feasible design can take as long as a full optimization. In the IPTRSAO framework Pérez *et al.*^{6,7}, the use of a two algorithm approach for the optimization was avoided by relaxing the constraints when an infeasible starting point is encountered. Probability one homotopy methods^{10,11} were used to relax the constraints to obtain a feasible design point within the trust region.

$$\begin{aligned} & \text{Choose } b_i > 0 \\ & gr_i(\mathbf{x}) = g_i(\mathbf{x}) + (1 - \tau)b_i \geq 0 \end{aligned} \quad (5)$$

Where b_i is a constant and τ drives the relaxed constraint gr_i to the actual constraint by gradually adjusting $\tau = 0 \rightarrow \tau = 1$.

The approximate minimization subproblem (2) can be solved with respect to the relaxed constraints.

$$\begin{aligned} & \min \quad \tilde{\theta} \\ & \text{s.t.} \quad \tilde{\mathbf{g}}\mathbf{r} \geq 0 \\ & \text{s.t.} \quad \tilde{\mathbf{h}}\mathbf{r} = 0 \\ & \mathbf{x}_L \leq \mathbf{x} \leq \mathbf{x}_U \end{aligned} \quad (6)$$

Note that the resulting point is feasible with respect to the relaxed inequality constraints. This is referred to as a *relaxed feasible point*. The most important characteristic of this algorithm is that all constraint violations are controlled by a single parameter, τ , and no single constraint dominates the optimization process. At each iteration, the parameter τ is gradually updated from $\tau = 0$ to $\tau = 1$. The optimization, has two steps, one, determining the value of τ for the next iteration, and the second step, in which the relaxed approximate minimization (6) is solved. To determine τ Pérez *et al.*⁷ present an heuristic approach that requires the solution of an approximate optimization to find the maximum value of τ that gives a relaxed feasible point within the trust region. In Pérez *et al.*⁶ a predictor-corrector scheme is implemented, using the machinery of the probability-one homotopy theory for nonlinear optimization Watson and Haftka¹¹, Watson *et al.*¹².

III. Composite step approaches

In trust region algorithms for the solution of nonlinear constrained optimization problems, variations of a composite step algorithm have been used to solve the infeasible trust region subproblem. Conn *et al.*³ offers an overview of the approaches. Two of them are of interest to our specific application. The Byrd-Omojokun-like approaches and the Celis-Dennis-Tapia-like approaches. The differences between these two and the homotopy approach described above lies in the formulation of the normal and tangential steps.

The composite step algorithms, have been used in SQP frameworks primarily, in which the objective function is a quadratic approximation of the Lagrangian function and the constraints are linearized. Here we presented a modified version of the composite step algorithm suited to a general SAO framework, where approximations to the objective function and constraints are readily available and the cost of solving an approximate minimization problem as (6) is small compared to the cost of function evaluations.

A. Normal step

The objective of the normal step, is to reduce the amount of constraint violation at each iteration. So the normal step is the solution of the problem

$$\begin{aligned} \min \quad & \| \min(0, \tilde{\mathbf{g}}(\mathbf{x}_0 + \mathbf{s}_n)) \|_2 \\ \text{s.t.} \quad & \|\mathbf{s}_n\| \leq \xi \Delta \end{aligned} \quad (7)$$

where $0 < \xi \leq 1$ limits the normal step to allow some freedom in the *tangential* step.

One could solve this problem exactly due to the availability of quadratic approximations to the constraints, i. e., find the model minimizer, or use an approximate solution approach as in Conn *et al.*³. The convergence theory only requires an approximate solution to this problem that guarantees some fraction of the Cauchy descent. So a quick solution is given by the Cauchy point:

$$\mathbf{s}^c = -\alpha^c \mathbf{A}^T \mathbf{g}(\mathbf{x}_0) \quad (8)$$

$$\begin{aligned} \alpha^c = \arg \min \quad & \| \tilde{\mathbf{g}}(\mathbf{x}_0 + \mathbf{s}^c(\alpha)) \|_2 \\ \text{s.t.} \quad & \|\mathbf{s}^c\| \leq \xi \Delta \end{aligned} \quad (9)$$

Note that although it is well known that the model minimizer used in unconstrained trust region methods is the most efficient in rate of convergence, in our application, the normal step only reduces the amount of constraint violation. It is expected that the algorithm may reach the feasible region some iterations before convergence of the algorithm is achieved, therefore both approaches (and any in-between) may have good performance.

B. Tangential step

The significant difference between the Byrd-Omojokun-like and the Celis-Dennis-Tapia-like approaches is in the formulation of the constraint violation once a normal step is taken.

1. Approach A: Byrd-Omojokun-like

The proposed approach is different to that in Pérez *et al.*⁶ in that the relaxation is not controlled by a single parameter for all constraints, but it is independent for each one. After computing the *normal* step \mathbf{s}_n that minimizes the constraint violation within a reduced trust region, a new formulation of the constraints take place:

$$gr_i(\mathbf{x}) = g_i(\mathbf{x}) - \min(0, g_i(\mathbf{x}_0 + \mathbf{s}_n)) \geq 0 \quad (10)$$

Note that the main difference between this and (5) is that the parameter τ controls the amount of relaxation for all constraints simultaneously, while in this approach the constraints can move independent of each other.

The optimization subproblem is solved as in (6).

2. Approach B: Celis-Dennis-Tapia-like

In this approach, the objective is to reduce the norm of the constraint violation, without looking at each constraint independently. Once the normal step \mathbf{s}_n has been computed, the norm of the constraint violation at the normal step is:

$$\kappa = \|\min(0, \tilde{\mathbf{g}}(\mathbf{x}_0 + \mathbf{s}_n))\|_2 \quad (11)$$

Then a modified form of the minimization subproblem (6) is solved:

$$\begin{aligned} & \min \quad \tilde{\theta} \\ \text{s.t.} \quad & \|\min(0, \tilde{\mathbf{g}}(\mathbf{x}))\| \leq \kappa \\ & \mathbf{x}_L \leq \mathbf{x} \leq \mathbf{x}_U \end{aligned} \quad (12)$$

This formulation allows some trade-off in the constraints. However, it is sensitive to the scaling of the constraints.

The combination of the two normal-step solutions and two tangential step formulations are to be compared with the homotopy approach of Pérez *et al.*⁷. Some heuristic decisions have to be done in choosing the value of ξ . A value close to one will move the design closer to the feasible region leaving small room for improvement of the objective function. Note that a value of $\xi = 1$ does have a more dramatic effect when the normal step is the model minimizer since it leaves no room for improvement of the objective function.

IV. Numerical experiments

To analyze the behavior of the different approaches, four test problems have been implemented within a trust region framework. Two of the problems have inequality constraints only, while two have both equality and inequality constraints. The four composite steps combinations plus the heuristic homotopy approach were implemented in the same Matlab©The Mathworks framework. Unless specified, the results are coded according to the type of composite step: *nh_n_t*, where $n = 0$ corresponds to the model minimizer and $n = 1$ is the Cauchy point. $t = 0$ corresponds to the Byrd-Omojokun-like tangential step, and $t = 1$ to the Celis-Dennis-Tapia-like tangential step.

A. Barnes problem

This is a simple nonlinear constrained problem with two design variables and three inequality constraints. A full description of the problem can be found at Pérez *et al.*⁷. The size of the problem and nonlinear nature helps to identify the behavior of the different options. The optimization results are shown in Table 1 in number of iterations required to converge. These results were obtained with $\xi = 0.9$.

Table 1. Number of iterations required to converge for the Barnes problem

Approach	P1	P2
homotopy	5	10
cs_exact_Byrd	5	11
cs_exact_Celis	5	9
cs_Cauchy_Byrd	5	10
cs_Cauchy_Celis	5	10

The results for this problem are very uniform for $\xi = 9$. In Figure 1 the number of iterations to converge vs. the value of ξ are shown. This plots show that the best results are obtained with $\xi = 0.9$. However, the results at other values are not so discouraging. At low values of ξ , the strategies seem to be more stable, giving the tangential step more freedom to reduce the value of the objective function. However from 0.4 to 0.9 fluctuations are more noticeable. These results are expected to be problem dependent. Figure 2 shows

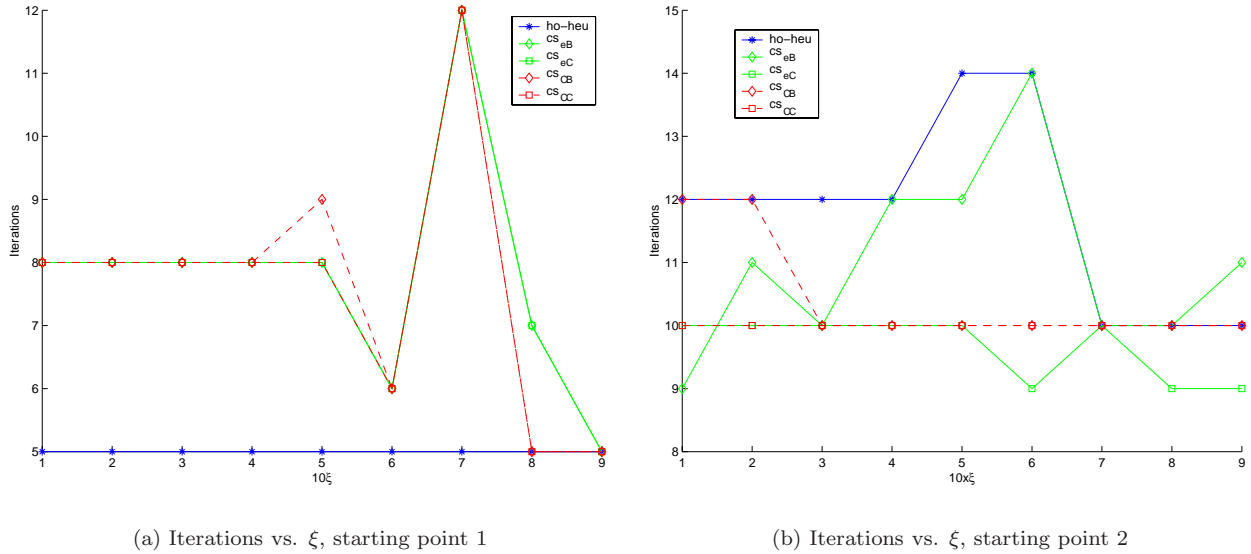


Figure 1. Convergence plots for different values of ξ

the effect of different values of ξ on the constraint violation. One can appreciate in this plot the different levels of allowed constraint violation for each iteration.

B. High performance, low-cost structure (HPLCS)

This problem consists of the design of a 10 bar structure. The objective is to minimize weight while maximizing load. The problem was introduced in Wujek *et al.*¹³ and consists of a total of 17 design variables (cross sections, trusses topology and payloads) and 13 inequality constraints. The results are shown on Table 2.

Table 2. Number of iterations required to converge for the HPLCS problem

Approach	P1
homotopy	26
cs_exact_Byrd	23
cs_exact_Celis	10
cs_Cauchy_Byrd	34
cs_Cauchy_Celis	9

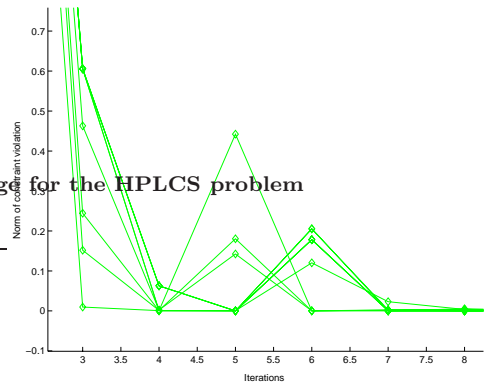


Figure 2. Constraint violation vs. iterations for different values of ξ . Case cs_exact_Celis

Plots of the history of the objective function, the norm of the constraint violation and the maximum constraint violation are presented in figure 3 on the following page. Note that both runs using the Celis-Dennis-Tapia-like tangential present early convergence. However the designs are feasible. The algorithm stopped because the size of the step was too small. This suggests that the Celis-Dennis-Tapia tangential step constrained the problem in such a way that only small steps were allowed. This problem

is known for having flat regions that allow early convergence.

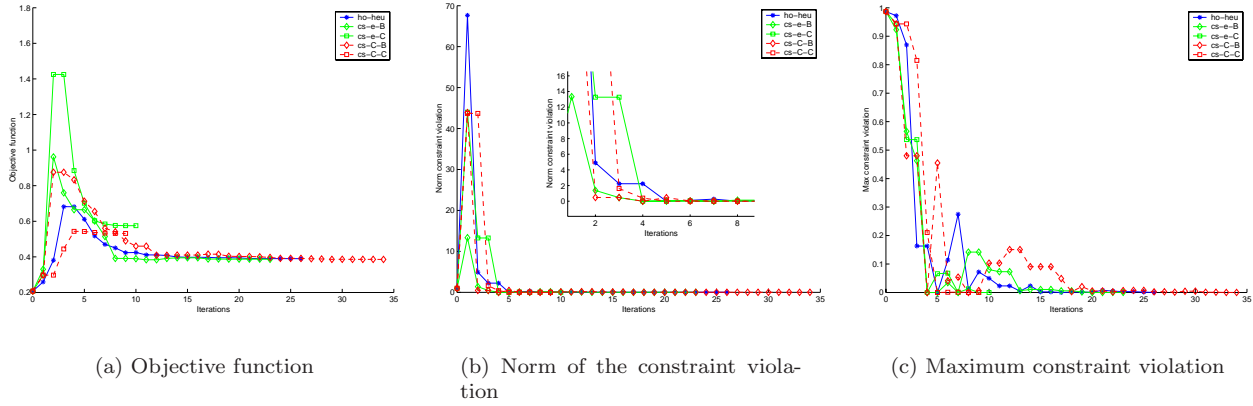


Figure 3. History plots for the HPLCS problem

C. Analytical reliability problem

This is a reliability-based design optimization problem. The equations of the objective function and constraints are analytical functions of the design variables. The problem contains equality constraints from the KKT equations of the reliability analysis. The problem and the formulation used in this analysis is presented in detail in Pérez *et al.*⁵. There are 6 design variables, 2 inequality constraints and 4 equality constraints. The results are presented in Table 3 and in Figure 4.

Table 3. Number of iterations required to converge for the analytical reliability problem

Approach	P1
homotopy	8
cs_exact_Byrd	11
cs_exact_Celis	14
cs_Cauchy_Byrd	13
cs_Cauchy_Celis	35

This type of problems is known to be highly non linear. In general, the homotopy heuristic step as well as the model minimizer normal step reduce the constraint violation faster than the Cauchy point. This is reflected on Figures 4 b and c. The combination of the Cauchy point as the normal step and the Celis-Dennis-Tapia tangential step had convergence problems, as it required 35 iterations to converge.

D. Beam reliability problem

This problem is similar in structure to the analytical reliability problem. There are 5 design variables, 1 inequality and 2 equality constraints. The equality constraints come from the KKT equations of the reliability analysis. A full description of the problem is found in Agarwal *et al.*¹. The results are presented in Table 4 and Figure 5.

The homotopy relaxation does a good job again. The use of the model minimizer for the normal step, independent of the tangential step, is better than using the Cauchy point as normal step. Note that in this problem the objective function is adjusted, not minimized from its starting point, making the normal step

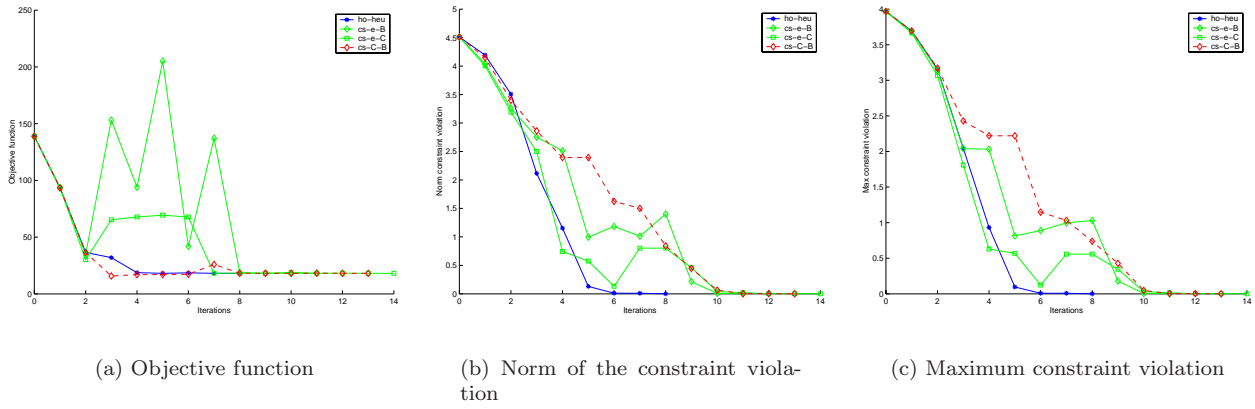


Figure 4. History plots for the analytical reliability problem

Table 4. Number of iterations required to converge for the beam reliability problem

Approach	P1
homotopy	19
cs_exact_Byrd	21
cs_exact_Celis	24
cs_Cauchy_Byrd	40
cs_Cauchy_Celis	18

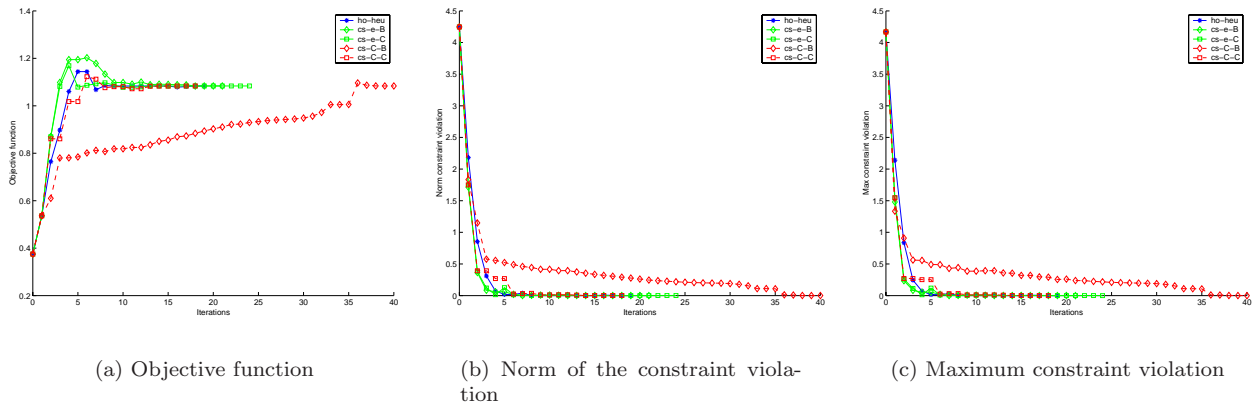


Figure 5. History plots for the beam reliability problem

more important until the design is feasible. The combination of Cauchy point with the Byrd-Omojokun tangential step converged slowly due to one of the equality constraints. As mentioned before, the dominance of a single constraint can be palliated by the homotopy approach. On the other hand, what proved to be a slow convergent approach in the previous problem, was very successful with this problem. This also underlines the problem-dependent nature of the approaches.

V. Concluding remarks

In this paper different techniques to control infeasible trust region subproblems for sequential approximate optimization are explored. Homotopy relaxation techniques are compared to composite-step approaches used in SQP-based nonlinear optimization algorithms. The composite-step techniques are adapted to the availability of cheap local approximations. Four test problems are used to analyze the behavior of the different techniques. The difference between the homotopy approach and the composite-step ones is that the first minimizes the maximum constraint violation and controls the amount of relaxation for all violated constraints at the same rate, while the normal step minimizes the norm of the constraint violation, allowing each constraint to relax independent of each other.

In spite of the reduced number of test problems, some observations can be drawn. When the normal step is solved using the model minimizer, the behavior in the presence of only inequality constraints is similar for the homotopy approach and the composite step. However, in the presence of equality constraints, the homotopy approach seems to perform better due to the simultaneous treatment of the relaxation of all constraints. The use of the Cauchy point, may decrease the performance of the algorithm, particularly in the presence of equality constraint. However, the use of some heuristics may palliate very bad cases. For the tangential step, the Byrd-Omojokun approach, similar to the one employed in the homotopy approach, seems to be more robust than the Celis-Denis-Tapia one.

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