INVESTIGATION OF RELIABILITY METHOD FORMULATIONS IN DAKOTA/UQ

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Abstract

Reliability methods for uncertainty quantification are probabilistic approaches that compute approximate response function distribution statistics based on specified uncertain variable probability distributions. In this paper, a variety of linearization approaches are explored for both the forward reliability analysis of computing probabilities for specified response levels (the reliability index approach (RIA)) and the inverse reliability analysis of computing response levels for specified probabilities (the performance measure approach (PMA)). Relative performance of these algorithmic variations are presented for a number of computational experiments performed using the DAKOTA/UQ software. Novel aspects include application of various linearization approaches to PMA, full cumulative distribution function mappings for level sequences, accurate warm starting of most probable point (MPP) searches using projections, and comparison of sequential quadratic programming and nonlinear interior-point optimization algorithms for the MPP searches. In addition, initial results are presented for the use of these RIA/PMA formulations for uncertainty quantification within bi-level reliability-based design optimization approaches.

1 Introduction

Reliability methods for uncertainty quantification are probabilistic approaches that compute approximate response function distribution statistics based on specified uncertain variable probability distributions. These response statistics include response mean, response standard deviation, and cumulative or complementary cumulative distribution function (CDF/CCDF) response-probability level pairs. These methods are often more efficient at computing statistics in the tails of the response distributions (events with low probability) than sampling-based approaches since the number of samples required to resolve a low probability can be prohibitive. Thus, these methods, as their name implies, are often used in a reliability context for assessing the probability of failure of a system when confronted with an uncertain environment.

This capability for assessing reliability is broadly useful within an optimization context, and reliability-based design optimization (RBDO) methods are popular approaches in industry for designing systems while accounting for variability and uncertainty. For example, corporate initiatives such as "design for six sigma" mesh well with RBDO capabilities.

In this paper, a variety of algorithms are explored for performing reliability analysis. In particular, forward and inverse reliability analyses are performed using multiple linearization, integration, warm starting, and optimization algorithm selections. These uncertainty quantification capabilities are then used as a foundation for exploring RBDO formulations. Sections 2-4 describe these algorithmic components, Section 5 provides computational results for two test problems, and Section 6 provides concluding remarks.

2 Reliability Method Formulations

2.1 Mean Value

The Mean Value method (MV, also known as MVFOSM in [1]) is the simplest, least-expensive reliability method in that it estimates the response means, response standard deviations, and all CDF/CCDF response-probability pairs from a single evaluation of response functions and their gradients at the uncertain variable means. This approximation can have acceptable accuracy when the response functions are nearly linear and their distributions are approximately Gaussian, but can have poor accuracy in other situations. The expressions for approximate response mean μ_g, approximate response standard deviation σ_g, response target to approximate probability/reliability level mapping ($\bar{z} \to p, \beta$), and probability/reliability target to approximate response level mapping ($\bar{p}, \bar{\beta} \to z$) are

$$\mu_g = g(\mu_{\mathbf{x}}) \quad (1)$$

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$$\sigma_g = \sum_i \sum_j Cov(i,j) \frac{dg}{dx_i}(\mu_{\mathbf{x}}) \frac{dg}{dx_j}(\mu_{\mathbf{x}}) \tag{2}$$

$$\beta_{cdf} = \frac{\mu_g - \bar{z}}{\sigma_g} \tag{3}$$

$$\beta_{ccdf} = \frac{\bar{z} - \mu_g}{\sigma_g} \tag{4}$$

$$z = \mu_g - \sigma_g \bar{\beta}_{cdf} \tag{5}$$

$$z = \mu_g + \sigma_g \bar{\beta}_{ccdf} \tag{6}$$

respectively, where $\mathbf{x}$ are the uncertain values in the space of the original uncertain variables ("x-space"), $g(\mathbf{x})$ is the "limit state" function (the response function for which probability-response level pairs are needed), and the CDF reliability index β_{cdf}, CCDF reliability index β_{ccdf}, CDF probability $p(g \leq z)$, and CCDF probability $p(g > z)$ are related to one another through

$$p(g \leq z) = \Phi(-\beta_{cdf}) \tag{7}$$

$$p(g > z) = \Phi(-\beta_{ccdf}) \tag{8}$$

$$\beta_{cdf} = -\Phi^{-1}(p(g \leq z)) \tag{9}$$

$$\beta_{ccdf} = -\Phi^{-1}(p(g > z)) \tag{10}$$

$$\beta_{cdf} = -\beta_{ccdf} \tag{11}$$

$$p(g \leq z) = 1 - p(g > z) \tag{12}$$

where $\Phi()$ is the standard normal cumulative distribution function. A common convention in the reliability literature is to define g in such a way that the CDF probability for a response level z of zero (i.e., $p(g \leq 0)$) is the response metric of interest. The formulations in this paper are not restricted to this convention and are designed to support CDF or CCDF mappings for general response, probability, and reliability level sequences.

2.2 MPP Search Methods

All other reliability methods solve a nonlinear optimization problem to compute a most probable point (MPP) and then integrate about this point (rather than the uncertain variable means as in MV) to compute probabilities. The MPP search is performed in transformed standard normal space ("u-space") since it simplifies the probability integration: the distance of the MPP from the origin has the meaning of the number of standard deviations separating the mean response from a particular response threshold. The transformation from x-space to u-space is performed using the transformation $u = T(x)$ with the reverse transformation denoted as $x = T^{-1}(u)$. Common approaches for performing these mappings include the Rosenblatt [2] and Nataf [3] transformations, where the results in this paper employ the latter.

The forward reliability analysis algorithm of computing CDF/CCDF probabilities for specified response levels is called the reliability index approach (RIA), and the inverse reliability analysis algorithm of computing response levels for specified CDF/CCDF probability levels is called the performance measure approach (PMA) [4]. The differences between the RIA and PMA formulations appear in the objective function and equality constraint formulations used in the MPP searches. For RIA, the MPP search for achieving the specified response level $\bar{z}$ is formulated as

$$\begin{array}{ll} \text{minimize} & \mathbf{u}^T\mathbf{u} \\ \text{subject to} & G(\mathbf{u}) = \bar{z} \end{array} \tag{13}$$

and for PMA, the MPP search for achieving the specified reliability/probability level $\bar{\beta}, \bar{p}$ is formulated as

$$\begin{array}{ll} \text{minimize} & \pm G(\mathbf{u}) \\ \text{subject to} & \mathbf{u}^T\mathbf{u} = \bar{\beta}^2 \end{array} \tag{14}$$

where $\mathbf{u}$ is a vector centered at the origin in u-space and $g(\mathbf{x}) \equiv G(\mathbf{u})$ by definition. In the RIA case, the optimal MPP solution $\mathbf{u}^*$ defines the reliability index from $\beta = \pm\|\mathbf{u}^*\|_2$, which in turn defines the CDF/CCDF probabilities (using Eqs. 7-8 in the case of first-order integration). The sign of β is defined by

$$G(\mathbf{u}^*) > G(\mathbf{0}) : \beta_{cdf} < 0, \beta_{ccdf} > 0 \tag{15}$$

$$G(\mathbf{u}^*) < G(\mathbf{0}) : \beta_{cdf} > 0, \beta_{ccdf} < 0 \tag{16}$$

where $G(\mathbf{0})$ is the median limit state response computed at the origin in u-space (where $\beta_{cdf} = \beta_{ccdf} = 0$ and first-order $p(g \leq z) = p(g > z) = 0.5$). In the PMA case, the sign applied to $G(\mathbf{u})$ (equivalent to minimizing or maximizing $G(\mathbf{u})$) is similarly defined by $\bar{\beta}$

$$\bar{\beta}_{cdf} < 0, \bar{\beta}_{ccdf} > 0 : \text{maximize } G(\mathbf{u}) \tag{17}$$

$$\bar{\beta}_{cdf} > 0, \bar{\beta}_{ccdf} < 0 : \text{minimize } G(\mathbf{u}) \tag{18}$$

and the limit state at the MPP ($G(\mathbf{u}^*)$) defines the desired response level result.

There are a variety of algorithmic variations that can be explored within RIA/PMA reliability analysis. First, one may select among several different linearization approaches for the limit state function that can be used to reduce computational expense during the MPP searches. Options include:

1. a single linearization per response/probability level in x-space centered at the uncertain variable means (commonly known as the Advanced Mean Value (AMV) method).

$$g(\mathbf{x}) \cong g(\mu_{\mathbf{x}}) + \nabla_x g(\mu_{\mathbf{x}})^T (\mathbf{x} - \mu_{\mathbf{x}}) \tag{19}$$

2. same as AMV, except that the linearization is performed in u-space. This option has been termed the u-space AMV method (note: $\mu_{\mathbf{u}} = T(\mu_{\mathbf{x}})$ and is nonzero in general).

$$G(\mathbf{u}) \cong G(\mu_{\mathbf{u}}) + \nabla_u G(\mu_{\mathbf{u}})^T (\mathbf{u} - \mu_{\mathbf{u}}) \tag{20}$$

3. an initial x-space linearization at the uncertain variable means, with iterative relinearizations at each MPP estimate ($\mathbf{x}^*$) until the MPP converges (commonly known as the AMV+ method).

$$g(\mathbf{x}) \cong g(\mathbf{x}^*) + \nabla_x g(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*) \tag{21}$$

4. same as AMV+, except that the linearizations are performed in u-space. This option has been termed the u-space AMV+ method.

$$G(\mathbf{u}) \cong G(\mathbf{u}^*) + \nabla_u G(\mathbf{u}^*)^T (\mathbf{u} - \mathbf{u}^*) \tag{22}$$

5. the MPP search on the original response functions without the use of any linearizations.

The second algorithmic variation involves the integration approach for computing probabilities at the MPP, which can be selected to be first-order (Eqs. 7-8) or second-order integration. Combining the no-linearization option of the MPP search with first-order and second-order integration approaches results in the traditional first-order and second-order reliability methods (FORM and SORM). Additional probability integration approaches can involve sampling in the vicinity of the MPP, and it is planned to support adaptive importance sampling for this purpose in the future.

Additional algorithmic variations include the optimization algorithm used to perform the MPP search and the use of warm starting approaches for improving computational efficiency. For the former, this paper explores the use of sequential quadratic programming (SQP) and nonlinear interior-point (NIP) optimization algorithms from the NPSOL [5] and OPT++ [6] libraries, respectively. And for the latter, a number of opportunities for the use of warm starts in reliability methods are detailed in the following section.

3 Warm Starting of MPP Searches

MPP searches can be accelerated through three distinct types of warm starting:

- with AMV+ iteration increment (UQ with AMV+)
- with $z/p/\beta$ level increment (UQ)
- with design variable change (RBDO)

and involve several different types of data:

- linearization point and associated response values (UQ with AMV+)
- MPP optimizer initial guess (UQ)

For subsequent $z/p/\beta$ levels in AMV+ approaches, the initial linearization point is warm started using the MPP from the previous level. The initial guess for the next MPP optimization is warm started using a simple copy of the previous MPP estimate in the case of unconverged AMV+ iterations or, in the case of an advance to the next $z/p/\beta$ level, by projecting from the current MPP out to the new β radius or response level. Note that premature optimization termination can occur if the RIA/PMA first-order optimality conditions ($\mathbf{u} + \lambda \nabla_u G = 0$ for RIA or $\nabla_u G + \lambda \mathbf{u} = 0$ for PMA) remain satisfied for the new level, even though the new equality constraint will be violated. The projection addresses this concern.

For the RIA case, compute an approximate $\mathbf{u}^{(k+1)}$ using a first-order Taylor series approximation of the next g level:

$$G^{(k+1)} = G^{(k)} + \nabla_u G(\mathbf{u}^{(k)})^T (\mathbf{u}^{(k+1)} - \mathbf{u}^{(k)}) \tag{23}$$

where $\mathbf{u}^{(k+1)}$ is defined as a projection along $\nabla_u G$ from $\mathbf{u}^{(k)}$

$$\mathbf{u}^{(k+1)} = \mathbf{u}^{(k)} + \alpha \nabla_u G(\mathbf{u}^{(k)}) \tag{24}$$

Substituting Eq. 24 into Eq. 23 defines the step

$$\alpha = \frac{G^{(k+1)} - G^{(k)}}{\|\nabla_u G(\mathbf{u}^{(k)})\|^2} \tag{25}$$

Note that this projection could bypass the need for $\nabla_u G$ with knowledge of the Lagrange multipliers at the current MPP ($\nabla_u G = -\frac{1}{\lambda}\mathbf{u}$ for RIA).

For the PMA case, compute an approximate $\mathbf{u}^{(k+1)}$ by scaling $\mathbf{u}^{(k)}$ so that its magnitude equals the next β target.

$$\mathbf{u}^{(k+1)} = \frac{\beta^{(k+1)}}{\beta^{(k)}} \mathbf{u}^{(k)} \tag{26}$$

In the case of multiple reliability method invocations within RBDO, the optimizer initial guess for the first level is warm started using the MPP for the first level from the previous reliability method invocation. For AMV+, the linearization point for the first level is also warm started using the previous MPP, although the response data at the linearization point must be reevaluated to account for design variable changes. Warm starts for all subsequent levels within the reliability analysis are performed as before. It is also possible to employ a projection for warm starting the optimizer initial guess across multiple reliability method invocations within RBDO [7]. The approach is similar to Eqs. 24-25 and will be included in future RBDO work.

4 Reliability-Based Design Optimization

Since an RBDO problem must specify both the z level and the p/β level, one can use either the RIA or the PMA formulation for the UQ portion and then constrain the result in the optimization portion. In particular, RIA uncertainty quantification maps z to p/β, so RIA RBDO constrains p/β:

$$\begin{aligned} \text{minimize} \quad & f \\ \text{subject to} \quad & \beta \geq \bar{\beta} \\ \text{or} \quad & p \leq \bar{p} \end{aligned} \tag{27}$$

And PMA uncertainty quantification maps p/β to z, so PMA RBDO constrains z:

$$\begin{aligned} \text{minimize} \quad & f \\ \text{subject to} \quad & z \geq \bar{z} \end{aligned} \tag{28}$$

where $z \geq \bar{z}$ is used as the RBDO constraint for a cumulative failure probability (failure defined as $z \leq \bar{z}$) but $z \leq \bar{z}$ would be used as the RBDO constraint for a complementary cumulative failure probability (failure defined as $z \geq \bar{z}$).

These approaches are known as bi-level RBDO since there are two distinct levels of optimization nested within each other, one at the design level and one at the MPP search level. Other approaches which seek to reduce the cost of nested optimizations include sequential, unilevel, and surrogate-based RBDO methods.

5 Computational Experiments

The algorithmic variations of interest in reliability analysis include the linearization approaches (MV, x/u-space AMV, x/u-space AMV+, and FORM), integration approaches (first/second-order), warm starting approaches, and MPP optimization algorithm selections (SQP or NIP). Bi-level RBDO algorithmic variations of interest include use of the RIA or PMA formulations for the underlying UQ as well as the specific $z/p/\beta$ mappings that are employed. Relative performance of these algorithmic variations will be demonstrated in this section for a number of computational experiments performed using the DAKOTA/UQ software [8]. DAKOTA/UQ is the uncertainty quantification component of DAKOTA [9], an open-source software framework for design and performance analysis of computational models on high performance computers.

Table 1: Reliability index approach results, lognormal ratio test problem.

RIA Approach	SQP Function Evals (Cold/Warm Start)	NIP Function Evals (Cold/Warm Start)	CDF p Error Norm	Target z Offset Norm
MV	5	5	0.2312	0.0
AMV	30	30	0.0521	0.5100
u-space AMV	30	30	0.0	0.6915
AMV+	631/541	631/541	0.0	0.0
u-space AMV+	521/401	521/401	0.0	0.0
FORM	1451/1116	661/491	0.0	0.0

Table 2: Performance measure approach results, lognormal ratio test problem.

PMA Approach	SQP Function Evals (Cold/Warm Start)	NIP Function Evals (Cold/Warm Start)	CDF z Error Norm	Target p Offset Norm
MV	5	5	0.6072	0.0
AMV	30	30	0.1166	0.0
u-space AMV	30	30	0.0	0.0
AMV+	526/481	561/481	0.0	0.0
u-space AMV+	246/246	286/251	0.0	0.0
FORM	3411/516	1002/861	0.0	0.0

5.1 Lognormal ratio

This test problem has a limit state function defined by the ratio of two lognormally-distributed random variables.

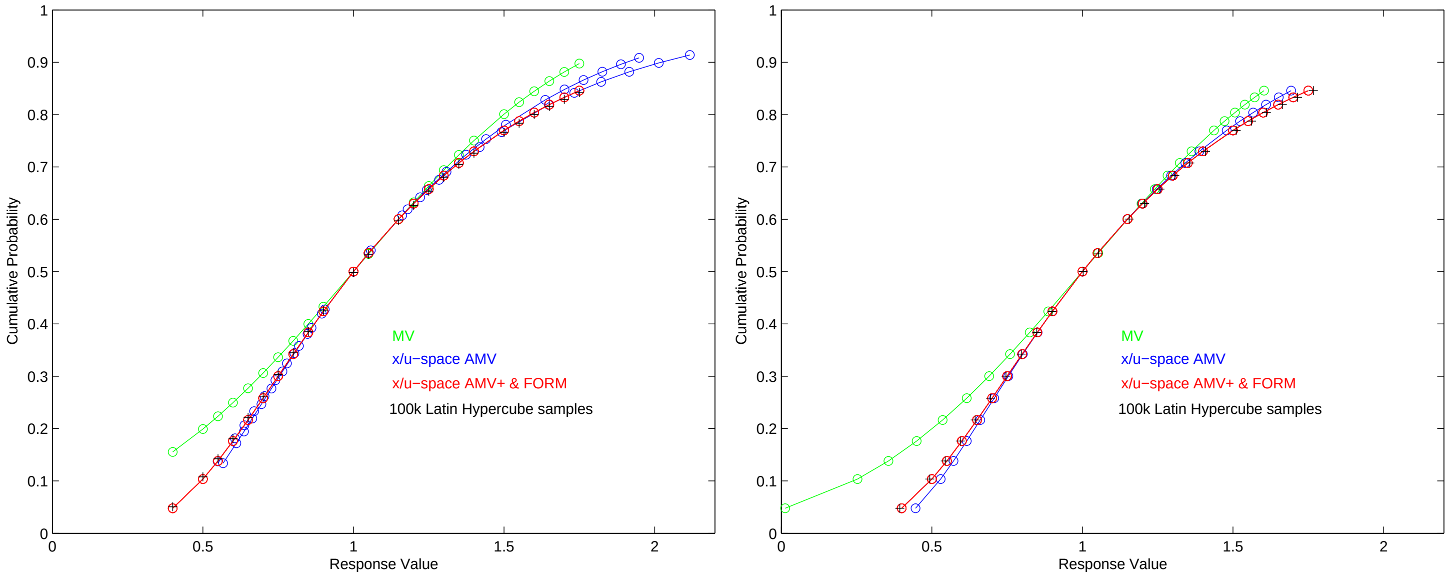
$$g(\mathbf{x}) = \frac{x_1}{x_2} \tag{29}$$

The distributions for both x_1 and x_2 are Lognormal(1, 0.5) with a correlation coefficient between the two variables of 0.3.

5.1.1 Uncertainty quantification

For RIA, a list of 24 response levels (.4, .5, .55, .6, .65, .7, .75, .8, .85, .9, 1, 1.05, 1.15, 1.2, 1.25, 1.3, 1.35, 1.4, 1.5, 1.55, 1.6, 1.65, 1.7, and 1.75) is mapped into the corresponding cumulative probability levels. For PMA, these 24 probability levels (the fully converged results from RIA FORM) are mapped back into the original response levels. Tables 1 and 2 show the computational results for each of the six method variants using numerical gradients computed with central differences. The RIA p error norms and PMA z error norms are measured relative to the fully-converged FORM results. That is, the FORM error (RIA p error norm of 0.0171 and PMA z error norm of 0.0397) relative to a Latin Hypercube reference solution of 100,000 samples is not included so as to avoid obscuring the relative errors. Figure 1 overlays the computed CDF values for each of the six method variants as well as the Latin Hypercube reference solution.

It is evident that, relative to the fully-converged AMV+/FORM results, MV accuracy degrades rapidly away from the means. AMV is reasonably accurate over the full range (x-space AMV has 4-5x reduction in error norm relative to MV and u-space AMV has zero error) but has undesirable offsets from the prescribed response levels in the RIA case. In terms of computational expense, MV is two orders of magnitude less expensive than AMV+/FORM and AMV is one order of magnitude less expensive, which makes these techniques attractive when rough statistics are sufficient. When more accurate statistics are desired, AMV+ has equal accuracy to FORM and is 2.3-14x less expensive in the case of cold starts using sequential quadratic programming (SQP) for each level, which decreases to a factor of 1.1-2.8x less expensive in the case of warm starts using SQP. That is, FORM benefits more from warm starting than AMV+. When using a nonlinear interior-point (NIP) optimizer, FORM solutions are generally less expensive and become directly competitive with AMV+ in some cases (AMV+ is 1.05-3.5x less expensive than FORM in the case of cold starts using NIP, which decreases slightly to a factor of 0.91-3.4x in the case of warm starts using NIP). The SQP/NIP comparison is much less relevant for the MV/AMV/AMV+ methods since the MPP searches are linearized. Another benefit of NIP relative to SQP has been observed in PMA solutions (Eq. 14). PMA solutions with SQP involve penalties applied to the equality constraint (e.g., in an augmented Lagrangian merit function) and must have strict u-space bound constraints (e.g., 10 standard deviations) to avoid excessive u-space excursions in minimizing $G(\mathbf{u})$ prior to enforcement of the $\mathbf{u}^T\mathbf{u}$ equality constraint. These excursions can result in inaccurate Hessian approximations in moderate cases and numerical overflow in extreme cases. NIP methods are less prone to this difficulty since they proceed toward constraint satisfaction more uniformly.



(a) RIA methods (b) PMA methods

Figure 1: Lognormal ratio CDF.

Table 3: Reliability index approach results, short column test problem.

RIA Approach	SQP Function Evals (Cold/Warm Start)	NIP Function Evals (Cold/Warm Start)	CDF p Error Norm	Target z Offset Norm
MV	1	1	0.1548	0.0
AMV	45	45	0.0093	18.28
u-space AMV	45	45	0.0064	18.81
AMV+	268/213	268/217	0.0	0.0
u-space AMV+	305/245	305/245	0.0	0.0
FORM	701/658	557/351	0.0	0.0

5.2 Short column

This test problem is a short column with rectangular cross section (width b and depth h) having uncertain material properties (yield stress P) and subject to uncertain loads (bending moment M and axial force Y) [10]. The limit state function is defined as:

$$g(\mathbf{x}) = 1 - \frac{4M}{bh^2Y} - \frac{P^2}{b^2h^2Y^2} \tag{30}$$

The distributions for P, M, and Y are Normal(500, 100), Normal(2000, 400), and Lognormal(5, 0.5), respectively, with a correlation coefficient of 0.5 between P and M (uncorrelated otherwise). The nominal values for b and h are 5 and 15, respectively.

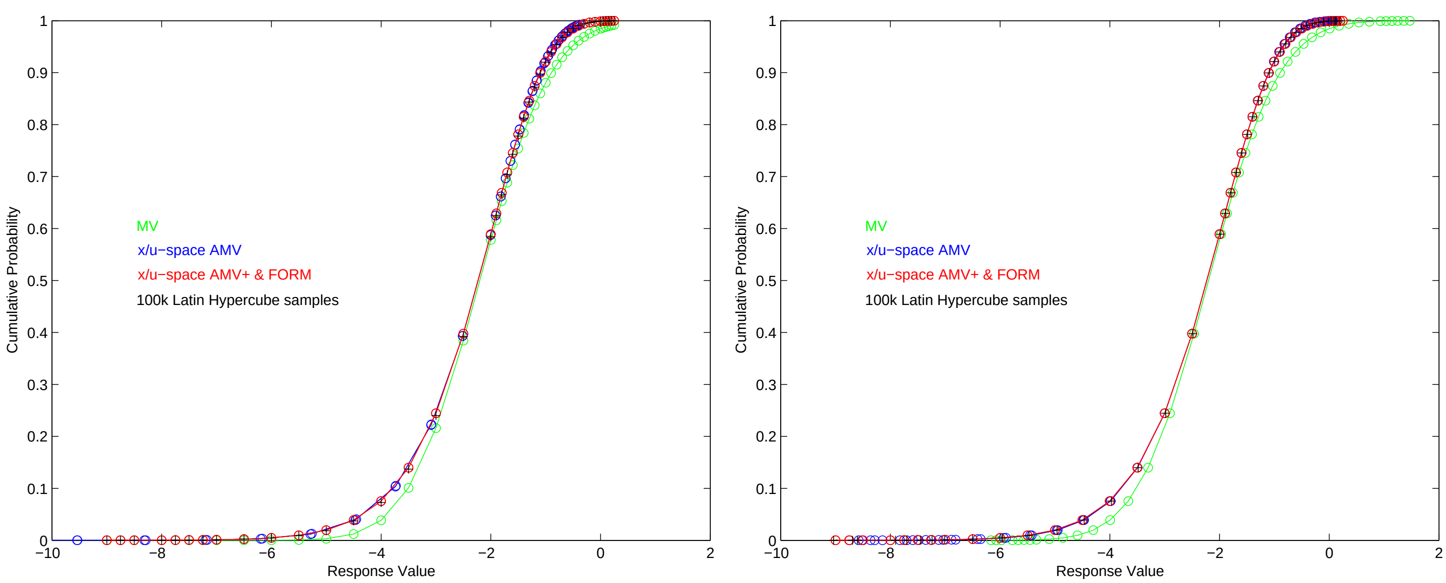
5.2.1 Uncertainty quantification

For RIA, a list of 43 response levels (-9.0, -8.75, -8.5, -8.0, -7.75, -7.5, -7.25, -7.0, -6.5, -6.0, -5.5, -5.0, -4.5, -4.0, -3.5, -3.0, -2.5, -2.0, -1.9, -1.8, -1.7, -1.6, -1.5, -1.4, -1.3, -1.2, -1.1, -1.0, -0.9, -0.8, -0.7, -0.6, -0.5, -0.4, -0.3, -0.2, -0.1, 0.0, 0.05, 0.1, 0.15, 0.2, 0.25) is mapped into the corresponding cumulative probability levels. For PMA, these 43 probability levels (the fully converged results from RIA AMV+/FORM) are mapped back into the original response levels. In this test problem, analytic gradients of f and g with respect to P, M, and Y are used to reduce function evaluation counts. Tables 3 and 4 show the computational results for each of the six method variants. The RIA p error norms and PMA z error norms are measured relative to the fully-converged FORM results. That is, the FORM error (RIA p error norm of 0.0138 and PMA z error norm of 0.2162) relative to a Latin Hypercube reference solution of 100,000 samples is not included so as to avoid obscuring the relative errors. Figure 2 overlays the computed CDF values for each of the six method variants as well as the Latin Hypercube reference solution.

Relative to the fully-converged AMV+/FORM results, MV accuracy again degrades rapidly away from the means. AMV is again reasonably accurate over the full range (8-24x reduction in error norm relative to MV) but has undesirable offsets

Table 4: Performance measure approach results, short column test problem.

PMA Approach	SQP Function Evals (Cold/Warm Start)	NIP Function Evals (Cold/Warm Start)	CDF z Error Norm	Target p Offset Norm
MV	1	1	7.454	0.0
AMV	45	45	0.9420	0.0
u-space AMV	45	45	0.5828	0.0
AMV+	237/206	221/216	0.0	0.0
u-space AMV+	287/243	273/246	0.0	0.0
FORM	1165/669	839/326	0.0	0.0



(a) RIA methods

(b) PMA methods

Figure 2: Short column CDF.

Table 5: Bi-level RBDO results, short column test problem.

RBDO Approach	Function Evals (Cold/Warm Start)	Objective Function	Constraint Violation
RIA $z \to p$ AMV+	420/382	217.1	0.0
RIA $z \to p$ FORM	2108/1901	217.1	0.0
RIA $z \to \beta$ AMV+	218/194	216.7	0.0
RIA $z \to \beta$ FORM	938/1104	216.7	0.0
PMA $p, \beta \to z$ AMV+	308/226	216.8	0.0
PMA $p, \beta \to z$ FORM	1523/745	216.8	0.0

from the prescribed response levels in the RIA case. In terms of computational expense, MV and AMV are again significantly less expensive. AMV+ has equal accuracy to FORM and is 2.3-4.9x less expensive than FORM in the case of cold starts using SQP for each level, which decreases to a factor of 2.7-3.2x in the case of warm starts using SQP. NIP-based FORM solutions are again less expensive than SQP-based FORM solutions and are approaching the expense of AMV+ solutions (AMV+ is 1.8-3.8x less expensive than FORM in the case of cold starts using NIP, which decreases to a factor of 1.3-1.6x in the case of warm starts using NIP).

5.2.2 Reliability-based design optimization

The short column example problem is also amenable to RBDO. Adding an objective function of cross-sectional area (minimizes weight in a uniform beam):

$$f(x) = bh \tag{31}$$

and a target reliability index of 2.5 (failure probability $= p(z \leq 0) \leq 0.00621$), Table 5 shows the results for bi-level RBDO using warm starts for the initial guess and AMV+ linearization points in the MPP searches. As is evident from the UQ results shown in Figure 2, the initial design of $(b, h) = (5, 15)$ is infeasible and the optimization must add material to obtain the target reliability at the optimal design $(b, h) = (8.68, 25.0)$.

It is important to note that only a single response/probability mapping is needed for each uncertainty analysis (instead of the 43 used previously). Analytic gradients of g with respect to P, M, and Y are used at the uncertainty analysis level, but numerical gradients of f and $z/p/\beta$ with respect to b and h are computed using central finite differences at the optimization level. SQP is used for optimization at both levels.

It is evident that applying reliability constraints using β is generally preferred to applying probability constraints using p in the RIA RBDO formulation of Eq. 27 (expense reduced by approximately a factor of 2), since β will tend to be more well-behaved/linear and well-scaled for the top-level optimizer than p. In addition, warm starts are generally helpful, typically saving 30% or more, and AMV+-based RBDO consistently outperforms FORM-based RBDO by a factor of 3.2-5.7x. No consistent preference for RIA-based or PMA-based RBDO is evident in this problem, although RIA AMV+ RBDO using warm starts and β constraints was the top performer and solved the problem in fewer than 200 function evaluations.

6 Conclusions

DAKOTA/UQ provides a flexible, object-oriented implementation of reliability methods that allows plug-and-play experimentation with RIA/PMA formulations and various linearization, integration, warm starting, and MPP optimization selections. In addition, these reliability methods provide a substantial foundation for exploring a variety of RBDO formulations. Novel aspects of this study have included (1) application of linearization approaches to PMA, (2) full CDF/CCDF mappings for level sequences, (3) accurate warm starting of MPP searches using projections, and (4) comparison of SQP and NIP optimization algorithms for MPP searches.

Reliability method performance comparisons for the two test problems presented indicate several trends. MV and AMV are significantly less expensive than AMV+ and FORM, but come with corresponding reductions in accuracy. AMV+, on the other hand, has equal accuracy to FORM and has been shown to have consistently significant computational savings (factor of 2.8 reduction in function evaluations on average). In addition, it appears that nonlinear interior-point optimizers may be both more robust (less susceptible to PMA u-space excursions) and more efficient (factor of 1.9 reduction in function evaluations on average for comparable accuracy in FORM solutions) than sequential quadratic programming optimizers for solution of most probable point searches. Of these two preferred approaches, AMV+ and FORM using nonlinear interior-point, AMV+ is the top performer in being a factor of 1.65 less expensive on average than FORM NIP.

Bi-level RBDO results mirror the uncertainty quantification trends. RBDO with AMV+ was shown to be a factor of 4.6 less expensive on average than RBDO with FORM. In addition, usage of β in reliability constraints was preferred due to it being more well-behaved and more well-scaled than constraints on p (resulting in a factor of 2 reduction in RIA RBDO expense). These bi-level RBDO results provide only a first step toward a production RBDO capability; the DAKOTA/UQ reliability methods will continue to provide a flexible foundation for exploring more advanced unilevel, sequential, and surrogate-based RBDO methods.

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